Journal of Statistical Physics, Vol. 16, No. 3, 1977

Analysis of Nonstationary, Gaussian and Non-Gaussian, Generalized Langevin Equations Using Methods of Multiplicative Stochastic Processes

Ronald Forrest Fox[1]

Received July 16, 1976

Using the methods of multiplicative stochastic processes, a thorough analysis of "non-Markovian," generalized Langevin equations is presented. For the Gaussian case, these methods are used to show that the nonstationary Fokker–Planck equation already found by Adelman and others is also obtainable from van Kampen's lemma for stochastic probability flows. Here, results applicable to an arbitrary n-component process are obtained and the specific two-component case of the Brownian harmonic oscillator is presented in detail in order to explicitly exhibit the matrix algebraic methods. The non-Gaussian case is presented at the end of the paper and shows that the methods already used in the Gaussian case lead directly to results for the non-Gaussian case. In order to use the methods of multiplicative stochastic processes analysis, it is necessary to transform the "non-Markovian," generalized Langevin equation using a stochastic extension of a transformation discussed by Adelman. This transformation removes the "memory kernel" term in the usual generalized Langevin equation and in the Gaussian case leads to the result that the original process was in fact not "non-Markovian" but actually nonstationary, *Markovian*.

KEY WORDS: Nonstationary; Gaussian; Markov process; generalized Langevin equation; multiplicative stochastic process.

1. INTRODUCTION

In this paper, the relationship between the generalized Langevin equation and the Fokker–Planck equation for the time evolution of the conditional probability distribution is elucidated. A method is exhibited which provides the results already obtained by Adelman[1] for the special case in which the

Supported through a fellowship from the Alfred P. Sloan Foundation.

[1] School of Physics, Georgia Institute of Technology, Atlanta, Georgia.

stochastic driving force is Gaussian. This method has the virtue that it directly leads to the corresponding results for the non-Gaussian case, and thereby provides the general and definitive treatment Adelman discussed in the summary of his paper.

The method to be applied in this paper utilizes ordered cumulants,[2,3] which were developed for the theory of multiplicative stochastic processes.[4] The generalized Langevin equation is an additive stochastic process and has been treated and derived with projection operator techniques.[5,6] Consequently, this paper will serve to illustrate the relationship between additive and multiplicative stochastic processes as well as to illustrate the alternative advantages of projection operator and cumulant techniques. While this last point has been discussed in the recent literature,[7,8] this paper provides a detailed account of the relative merits of the methods in the explicit solution to the problem of finding the conditional probability distribution for generalized Langevin equations.

Two lemmas will be presented which are necessary in order to implement the program presented in this paper. The first lemma follows from the discussion in Section V of Adelman's paper,[1] and shows that the "memory kernel" of the generalized Langevin equation can always be transformed into a term without "memory." The second lemma was proved by van Kampen[9] and involves stochastic probability flows, which were discussed somewhat earlier by Fox,[10] but without the proof provided by van Kampen.[2]

In Section 2 of this paper, the generalized Langevin equation will be presented and briefly reviewed. A stochastic generalization of the Adelman transformation will be introduced in order to eliminate the "memory kernel." In Section 3, a related stochastic probability flow will be introduced, and van Kampen's lemma will be used to show that the stochastic average of the probability flow leads directly to the conditional probability distribution equation, which will be referred to as the Fokker–Planck equation. In Section 4, a second example, the Brownian oscillator, will be presented in order to exhibit the fact that the methods used apply equally well to multicomponent equations. In Section 5, a discussion of the non-Gaussian generalization is given, and provides the solution Adelman[1] desired.

2. GENERALIZED LANGEVIN EQUATION

The generalized Langevin equation provides a "non-Markovian" extension of Langevin's equation for Brownian motion. The equation is, in one dimension,

$$\frac{d}{dt} u(t) = -\int_0^t \beta(t - s) u(s)\, ds + \frac{1}{m} \tilde{f}(t) \tag{1}$$

[2] See references in Fox[10] to Ryogo Kubo, who used these ideas earliest, but also without proof.

in which $u(t)$ is the velocity of the Brownian particle at time t, m is its mass, $\beta(t - s)$ is the dissipative "memory kernel," and $\tilde{f}(t)$ is the stochastic driving force. $\tilde{f}(t)$ is assumed to have zero mean, which is denoted by $\langle \tilde{f}(t) \rangle = 0$, and has a variance given by[6,11]

$$\langle \tilde{f}(t)\tilde{f}(s) \rangle = K_B T m \beta(t - s) \tag{2}$$

in which K_B is Boltzmann's constant and T is the temperature. In order to specify the higher order moments of $\tilde{f}(t)$, it is necessary to be more specific about what kind of process $\tilde{f}(t)$ actually is. Usually, it is assumed to be Gaussian, although this is never proved on the basis of a truly microscopic theory. If $\beta(t - s) = 2\beta\delta(t - s)$, in which $\delta(t)$ denotes the Dirac delta function, then (1) and (2) correspond to the Markovian Langevin equation.[12] The restriction to one dimension is not necessary and is made here in order to most clearly exhibit the nature of the mathematics, and in Section 4 a two-component equation will be treated to illustrate the extension to multi-component cases. Equation (2) is called the fluctuation–dissipation relation because it couples the variance of the stochastic force to the strength of the dissipative memory kernel.

The solution to (1) is obtained by introducing the Laplace transform of $\beta(\tau)$, which is denoted by $\hat{\beta}(z)$ and defined by

$$\hat{\beta}(z) = \int_0^\infty e^{-z\tau}\beta(\tau)\, d\tau \tag{3}$$

Using $\chi(t)$, which is defined through its Laplace transform, which is

$$\hat{\chi}(z) = [z + \hat{\beta}(z)]^{-1} \tag{4}$$

we obtain the solution for $u(t)$ as

$$u(t) = \chi(t)u(0) + \frac{1}{m}\int_0^t \chi(t - s)\tilde{f}(s)\, ds \qquad \text{and} \qquad \chi(0) = 1 \tag{5}$$

All of this is well known and has appeared in the literature many times.

By stochastically extending a transformation introduced by Adelman,[1] it is possible to convert (1) into an equivalent equation without a memory kernel. From (5) it follows that

$$u(0) = \frac{1}{\chi(t)}\left[u(t) - \frac{1}{m}\int_0^t \chi(t - s)\tilde{f}(s)\, ds\right] \tag{6}$$

and that

$$\frac{d}{dt}u(t) = \frac{d}{dt}\chi(t)u(0) + \frac{1}{m}\frac{d}{dt}\int_0^t \chi(t - s)\tilde{f}(s)\, ds \tag{7}$$

Using (6) in (7), we obtain

$$\frac{d}{dt}u(t) = \frac{(d/dt)\chi(t)}{\chi(t)}u(t) + \frac{1}{m}\chi(t)\frac{d}{dt}\int_0^t \frac{\chi(t-s)}{\chi(t)}\tilde{f}(s)\,ds$$
$$= -\bar{\beta}(t)u(t) + \tilde{g}(t) \tag{8}$$

in which the second equality defines both $\bar{\beta}(t)$ and $\tilde{g}(t)$. Note that $\langle\tilde{g}(t)\rangle = 0$. This transformation will be called Adelman's lemma.

Two points are worth emphasizing at this stage. Because Eq. (8) is a single time equation, without a memory kernel, it may be suspected that the original "non-Markovian" process is actually Markovian after all. This point will be expanded upon later, and at the end of the paper. Second, the objection may be raised that $\chi(t)$ can become zero and then negative,[13] so that division by $\chi(t)$ when it is zero is not defined. Note, however, that if the average of (8) is considered

$$\frac{d}{dt}\langle u(t)\rangle = \left[\frac{d}{dt}\ln\chi(t)\right]\langle u(t)\rangle \tag{9}$$

then its solution is

$$\langle u(t)\rangle = \exp\left\{\int_0^t\left[\frac{d}{ds}\ln\chi(s)\right]ds\right\}\langle u(0)\rangle = \chi(t)\langle u(0)\rangle \tag{10}$$

which agrees with (5) and demonstrates that the division by zero does not in fact introduce extraneous behavior. Later, it will be seen that the variance of $u(t)$ also behaves correctly even if $\chi(t)$ is allowed to become negative.

In Adelman's[1] use of the transformation of (1) into (8), no discussion of the stochastic driving force $\tilde{g}(t)$ appeared. Here, it will be seen to be of great importance in the subsequent sections.

3. STOCHASTIC PROBABILITY FLOWS

Associated with Eq. (8) and the initial condition $u(t=0) = u(0)$ is the conditional probability distribution $P(u, t)$, which is conditioned by the initial condition: $P(u, 0) = \delta(u - u(0))$. Also associated with (8) is a "phase space" description in which $\rho(u, t)$ denotes a density of phase space points at time t determined from the initial density $\rho(u, 0)$ by the equation of motion (8). Because this phase space density is conserved, it satisfies a continuity equation, which is

$$\frac{\partial}{\partial t}\rho(u, t) = -\frac{\partial}{\partial u}[\dot{u}\rho(u, t)] \tag{11}$$

in which $\dot{u}$ denotes $(d/dt)u$ and which becomes

$$\frac{\partial}{\partial t}\rho(u,t) = -\frac{\partial}{\partial u}\{[-\bar{\beta}(t)u + \tilde{g}(t)]\rho(u,t)\} \tag{12}$$

when (8) is used. The presence of $\tilde{g}(t)$ in (12) makes (1) a multiplicative stochastic process[4,9] and ordered cumulant methods may be used to determine $\langle\rho(u,t)\rangle$. van Kampen[9] has proved that under the conditions stipulated above, $\langle\rho(u,t)\rangle \equiv P(u,t)$. This will be called van Kampen's lemma.

The utility of van Kampen's lemma is that it provides a method for obtaining the conditional probability distribution for an equation such as (8), or equivalently, it enables one to obtain the corresponding Fokker–Planck equation. In order to use this lemma, however, it was essential that the Adelman transformation be performed in order that the "non-Markovian," generalized Langevin equation be in a form suitable for application of multiplicative stochastic process methods.

The following, somewhat lengthy, computations will provide the Fokker–Planck equation for $\langle\rho(u,t)\rangle \equiv P(u,t)$. In order to make the presentation as intelligible as possible, it will initially be assumed that $\tilde{f}(t)$ in (1) is Gaussian. As a consequence, the result obtained for $P(u,t)$ will be identical with the result already presented by Adelman and others.[1] However, in Section 5 the non-Gaussian extension will be elucidated.

Define $\phi(u,t)$ by the "interaction picture"

$$\rho(u,t) \equiv \left\{\exp\left[-\frac{\partial}{\partial u}u\int_0^t \frac{\dot{\chi}(s)}{\chi(s)}\,ds\right]\right\}\phi(u,t) = \left\{\exp\left[-\frac{\partial}{\partial u}u\ln\chi(t)\right]\right\}\phi(u,t) \tag{13}$$

$\phi(u,t)$ satisfies

$$\frac{\partial}{\partial t}\phi(u,t) = -\left\{\exp\left[\frac{\partial}{\partial u}u\ln\chi(t)\right]\right\}\frac{\partial}{\partial u}\tilde{g}(t)\left\{\exp\left[-\frac{\partial}{\partial u}u\ln\chi(t)\right]\right\}\phi(u,t) \tag{14}$$

Identity 1:

$$\begin{aligned}\exp\left[\frac{\partial}{\partial u}u\ln\chi(t)\right]&\frac{\partial}{\partial u}\tilde{g}(t)\exp\left[-\frac{\partial}{\partial u}u\ln\chi(t)\right]\\ &= \frac{1}{m}\frac{d}{dt}\int_0^t \frac{\chi(t-s)}{\chi(t)}\tilde{f}(s)\,ds\frac{\partial}{\partial u}\end{aligned} \tag{15}$$

Proof:

$$\exp\left[\frac{\partial}{\partial u}u\ln\chi(t)\right]\frac{\partial}{\partial u}\tilde{g}(t)\exp\left[-\frac{\partial}{\partial u}u\ln\chi(t)\right]$$

$$
\begin{aligned}
&= \exp\left\{\ln \chi(t)\left[\frac{\partial}{\partial u} u, \cdot\right]\right\} \frac{\partial}{\partial u} \tilde{g}(t) \\
&= \tilde{g}(t) \sum_{n=0}^{\infty} \frac{1}{n!} [\ln \chi(t)]^n \left[\frac{\partial}{\partial u} u, \cdot\right]^n \frac{\partial}{\partial u} \\
&= \tilde{g}(t) \sum_{n=0}^{\infty} \frac{1}{n!} [\ln \chi(t)]^n (-1)^n \frac{\partial}{\partial u} \\
&= \tilde{g}(t) \exp[-\ln \chi(t)] \frac{\partial}{\partial u} \\
&= \tilde{g}(t) \frac{1}{\chi(t)} \frac{\partial}{\partial u} \\
&= \frac{1}{m} \frac{d}{dt} \int_0^t \frac{\chi(t-s)}{\chi(t)} \tilde{f}(s)\, ds \frac{\partial}{\partial u}
\end{aligned} \tag{16}
$$

In line three of the proof, $[(\partial/\partial u)u, \cdot]^n$ denotes the nth power or iteration of the commutator operator $[(\partial/\partial u)u, \cdot]$, which is to act on the operator $\partial/\partial u$ to the right. It is easily verified that $[(\partial/\partial u)u, \cdot]^n\, \partial/\partial u = (-1)^n\, \partial/\partial u$, which has been used to get from line three to line four. The last line follows from (8).

Therefore, (14) becomes

$$
\frac{\partial}{\partial t} \phi(u, t) = -\frac{1}{m} \frac{d}{dt} \int_0^t \frac{\chi(t-s)}{\chi(t)} \tilde{f}(s)\, ds \frac{\partial}{\partial u} \phi(u, t) \tag{17}
$$

Because $\langle \tilde{f}(t) \rangle = 0$ for all t, and because $\tilde{f}(t)$ is assumed to be Gaussian at this stage, ordered cumulant methods[4] provide an exact result involving only the second cumulant, which gives for the average of $\phi(u, t)$

$$
\begin{aligned}
\frac{\partial}{\partial t} \langle \phi(u, t) \rangle = &\int_0^t ds \frac{d}{dt} \int_0^t dt' \frac{d}{ds} \int_0^s ds' \\
&\times \frac{\chi(t-t')}{\chi(t)} \frac{\chi(s-s')}{\chi(s)} \frac{K_B T}{m} \beta(t'-s') \frac{\partial^2}{\partial u^2} \langle \phi(u, t) \rangle
\end{aligned} \tag{18}
$$

In fact, for this simple case, the differential operator in (17) commutes with itself at different times, so that ordinary cumulants are adequate. If the differential operator in (17) had been noncommuting at different times, then even its Gaussianness would not have permitted an exact expression involving only the second cumulant.[4] Higher order, ordered cumulants would have been required, and, in fact, are going to be discussed in Section 5 when the non-Gaussian case is discussed because non-Gaussianness also requires higher order cumulants.

In (18) the s-integration involves an integrand that is an exact differential, so that

$$\begin{aligned}
&\int_0^t ds \frac{d}{dt} \int_0^t dt' \frac{d}{ds} \int_0^s ds' \frac{\chi(t-t')}{\chi(t)} \frac{\chi(s-s')}{\chi(s)} \frac{K_B T}{m} \beta(t'-s') \\
&\quad = \frac{K_B T}{m} \int_0^t ds' \frac{\chi(t-s')}{\chi(t)} \frac{d}{dt} \int_0^t dt' \frac{\chi(t-t')}{\chi(t)} \beta(t'-s') \\
&\quad = \frac{1}{2} \frac{K_B T}{m} \frac{d}{dt} \int_0^t ds' \int_0^t dt' \frac{\chi(t-s')}{\chi(t)} \frac{\chi(t-t')}{\chi(t)} \beta(t'-s') \\
&\quad = \frac{1}{2} \frac{K_B T}{m} \frac{d}{dt} [\chi^{-2}(t) A(t)]
\end{aligned} \tag{19}$$

where the last line defines $A(t)$, which can be rewritten as

$$A(t) = \int_0^t d\tau \int_0^t d\sigma\, \chi(\tau)\beta(\sigma - \tau)\chi(\sigma) \tag{20}$$

Adelman[1] has shown that

$$\frac{d}{dt} A(t) = 2\chi(t) \int_0^t d\sigma\, \beta(\sigma - t)\chi(\sigma) = -2\chi(t)\dot{\chi}(t) \tag{21}$$

because the Laplace transforms of $\dot{\chi}(t)$ and $\int_0^t d\sigma\, \chi(\sigma)\beta(\sigma - t)$ are, respectively, $z\hat{\chi}(z) - 1$ and $\hat{\chi}(z)\hat{\beta}(z) \equiv 1 - z\hat{\chi}(z)$ because (4) implies $\hat{\beta}(z) = \hat{\chi}^{-1}(z) - z$. The solution to (21) is

$$A(t) = 1 - \chi^2(t) \tag{22}$$

which is compatible with $\chi(0) = 1$ and (20). Therefore,

$$\frac{d}{dt} [\chi^{-2}(t)A(t)] = -2 \frac{\dot{\chi}(t)}{\chi^3(t)} \tag{23}$$

Putting (23) into (19) and then into (18), we obtain

$$\frac{\partial}{\partial t} \langle \phi(u, t) \rangle = -\frac{K_B T}{m} \frac{\dot{\chi}(t)}{\chi^3(t)} \frac{\partial^2}{\partial u^2} \langle \phi(u, t) \rangle \tag{24}$$

Returning to (13), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \rho(u, t) \rangle &= -\frac{\dot{\chi}(t)}{\chi(t)} \frac{\partial}{\partial u} [u \langle \rho(u, t) \rangle] + \exp\left[-\frac{\partial}{\partial u} u \ln \chi(t)\right] \frac{\partial}{\partial t} \langle \phi(u, t) \rangle \\
&= -\frac{\dot{\chi}(t)}{\chi(t)} \frac{\partial}{\partial u} [u \langle \rho(u, t) \rangle] - \frac{K_B T}{m} \frac{\dot{\chi}(t)}{\chi^3(t)} \exp\left[-\frac{\partial}{\partial u} u \ln \chi(t)\right] \\
&\quad \times \frac{\partial^2}{\partial u^2} \left\{ \exp\left[+\frac{\partial}{\partial u} u \ln \chi(t)\right] \right\} \langle \rho(u, t) \rangle
\end{aligned} \tag{25}$$

Identity 2:

$$\exp\left[-\frac{\partial}{\partial u} u \ln \chi(t)\right] \frac{\partial^2}{\partial u^2} \exp\left[\frac{\partial}{\partial u} u \ln \chi(t)\right] = \chi^2(t) \frac{\partial^2}{\partial u^2} \tag{26}$$

Proof:

$$\begin{aligned}
&\exp\left[-\frac{\partial}{\partial u} u \ln \chi(t)\right] \frac{\partial^2}{\partial u^2} \exp\left[\frac{\partial}{\partial u} u \ln \chi(t)\right] \\
&= \exp\left\{-\ln \chi(t)\left[\frac{\partial}{\partial u} u, \cdot\right]\right\} \frac{\partial^2}{\partial u^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} [\ln \chi(t)]^n \left[\frac{\partial}{\partial u} u, \cdot\right]^n \frac{\partial^2}{\partial u^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} [\ln \chi(t)]^n (-2)^n \frac{\partial^2}{\partial u^2} = \chi^2(t) \frac{\partial^2}{\partial u^2}
\end{aligned} \tag{27}$$

To get line four of (27), $[(\partial/\partial u)u, \cdot]^n \, \partial^2/\partial u^2 = (-2)^n \, \partial^2/\partial u^2$ was used and is easily verified.

Consequently, Eq. (25) becomes

$$\frac{\partial}{\partial t} \langle \rho(u, t)\rangle = -\frac{\dot{\chi}(t)}{\chi(t)} \frac{\partial}{\partial u} [u\langle \rho(u, t)\rangle] - \frac{K_B T}{m} \frac{\dot{\chi}(t)}{\chi(t)} \frac{\partial^2}{\partial u^2} \langle \rho(u, t)\rangle \tag{28}$$

or equivalently, using (8) and van Kampen's lemma, we obtain

$$\frac{\partial}{\partial t} P(u, t) = \bar{\beta}(t) \frac{\partial}{\partial u} [uP(u, t)] + \frac{K_B T}{m} \bar{\beta}(t) \frac{\partial^2}{\partial u^2} P(u, t) \tag{29}$$

which is a Fokker-Planck equation for $P(u, t)$.

Especially notice that (29) contains t-dependent coefficients in $\bar{\beta}(t)$. This means that (29) describes a nonstationary, Gaussian, Markov process because (29) is a nonstationary diffusion equation,[15] and its explicit solution, which is given below, can be shown to satisfy the nonstationary Chapman-Kolmogorov equation.[15] Another way to express this is to note that (29) is Kolmogorov's forward equation[15] for a nonstationary Markov process in which the diffusion coefficient is t dependent but not u dependent.

By direct substitution into (29) it can be seen that the solution to (29) with $P(u, 0) = \delta(u - u(0))$ is given by the nonstationary Gaussian conditional probability distribution

$$P(u, t) = [2\pi\sigma^2(t)]^{-1/2} \exp\left[-\frac{[u - \chi(t)u(0)]^2}{2\sigma^2(t)}\right] \tag{30}$$

in which $\sigma^2(t)$ satisfies the equation

$$\frac{1}{2} \frac{d}{dt} \sigma^2(t) = \frac{\dot{\chi}(t)}{\chi(t)} \left[\sigma^2(t) - \frac{K_B T}{m}\right] \tag{31}$$

with initial condition $\sigma^2(0) = 0$. The solution to (31) is

$$\sigma^2(t) = \frac{K_B T}{m}[1 - \chi^2(t)] = \frac{K_B T}{m} A(t) \tag{32}$$

Equations (30) and (32) provide the complete stochastic description of the solution to (1) in the Gaussian case. Notice also that if $\chi(t)$ becomes negative, as discussed earlier, no difficulty arises.

Before proceeding to the non-Gaussian case, which is given in Section 5, a two-component case will be presented to illustrate the fact that none of the preceding considerations are limited to one-component equations. These considerations will occupy Section 4.

4. "NON-MARKOVIAN" BROWNIAN OSCILLATOR

The traditional example for a multicomponent Langevin equation is the Brownian motion of a harmonic oscillator.[12,14] If ω denotes the unperturbed frequency of the oscillator, m its mass, p its momentum, and $y = m\omega x$, where x is its position, then the equations of motion can be written in terms of a two-component vector:

$$\frac{d}{dt}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} y \\ p \end{pmatrix} - \int_0^t \begin{pmatrix} 0 & 0 \\ 0 & \beta(t-s) \end{pmatrix}\begin{pmatrix} y(s) \\ p(s) \end{pmatrix} ds + \begin{pmatrix} 0 \\ \tilde{f}(t) \end{pmatrix} \tag{33}$$

in which $\beta(t - s)$ is the memory kernel and $\tilde{f}(t)$ is the stochastic driving force. As in the example of Sections 2 and 3, $\langle \tilde{f}(t)\rangle = 0$ and $\langle \tilde{f}(t)\tilde{f}(s)\rangle = K_B T m\beta(t - s)$. It will also be assumed in this section that $\tilde{f}(t)$ is Gaussian. The non-Gaussian case will be discussed in Section 5. Using Laplace transforms, we obtain

$$\begin{pmatrix} \hat{y}(z) \\ \hat{p}(z) \end{pmatrix} = \left[z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \hat{\beta}(z) \end{pmatrix}\right]^{-1}\left[\begin{pmatrix} y(0) \\ p(0) \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{f}(z) \end{pmatrix}\right] \tag{34}$$

It is easily verified using the matrix adjoint method of construction that the inverse of the matrix in (34) is given by

$$\begin{pmatrix} z & -\omega \\ +\omega & z + \hat{\beta}(z) \end{pmatrix}^{-1} = \begin{pmatrix} z + \hat{\beta}(z) & \omega \\ -\omega & z \end{pmatrix} \frac{1}{z[z + \hat{\beta}(z)] + \omega^2} \tag{35}$$

Let $\chi_p(t)$ be defined through its Laplace transform by

$$\hat{\chi}_p(z) \equiv [z^2 + z\hat{\beta}(z) + \omega^2]^{-1} \tag{36}$$

Because in this case $\chi_p(0) = 0$, $z\hat{\chi}_p(z)$ is the Laplace transform of $\dot{\chi}_p(t)$.

Therefore, (34) can be inverse-Laplace-transformed into

$$\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \dot{\chi}_p(t) + \int_0^t \beta(t-s)\chi_p(s)ds & \omega\chi_p(t) \\ -\omega\chi_p(t) & \dot{\chi}_p(t) \end{pmatrix} \begin{pmatrix} y(0) \\ p(0) \end{pmatrix}$$
$$+ \begin{pmatrix} \omega \int_0^t \chi_p(t-s)\tilde{f}(s)\,ds \\ \int_0^t \dot{\chi}_p(t-s)\tilde{f}(s)\,ds \end{pmatrix} \tag{37}$$

It also follows that $\dot{\chi}_p(0) = 1$, which guarantees the initial value requirements. Equations (33) and (37) also imply $\ddot{\chi}_p(0) = 0$.

To perform the Adelman transformation on Eq. (33), it is convenient to rewrite (37) as

$$\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} \equiv \mathbf{M}(t)\begin{pmatrix} y(0) \\ p(0) \end{pmatrix} + \begin{pmatrix} \tilde{F}_y(t) \\ \tilde{F}_p(t) \end{pmatrix} \tag{38}$$

which defines $\mathbf{M}(t)$ and $\tilde{F}_y(t)$ and $\tilde{F}_p(t)$. Therefore

$$\begin{pmatrix} y(0) \\ p(0) \end{pmatrix} = \mathbf{M}^{-1}(t)\left[\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} - \begin{pmatrix} \tilde{F}_y(t) \\ \tilde{F}_p(t) \end{pmatrix}\right] \tag{39}$$

and

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} = \dot{\mathbf{M}}(t)\begin{pmatrix} y(0) \\ p(0) \end{pmatrix} + \frac{d}{dt}\begin{pmatrix} F_y(t) \\ F_p(t) \end{pmatrix} \tag{40}$$

Substitution of (39) into (40) gives

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} = \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} - \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\begin{pmatrix} \tilde{F}_y(t) \\ \tilde{F}_p(t) \end{pmatrix} + \frac{d}{dt}\begin{pmatrix} \tilde{F}_y(t) \\ \tilde{F}_p(t) \end{pmatrix}$$
$$\equiv \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} + \begin{pmatrix} \tilde{G}_y(t) \\ \tilde{G}_p(t) \end{pmatrix} \tag{41}$$

in which the last line defines $\tilde{G}_y(t)$ and $\tilde{G}_p(t)$. An explicit calculation shows that

$$\begin{pmatrix} \tilde{G}_y(t) \\ \tilde{G}_p(t) \end{pmatrix} = \mathbf{M}(t)\frac{d}{dt}\int_0^t \mathbf{M}^{-1}(t)\mathbf{M}(t-s)\begin{pmatrix} 0 \\ \tilde{f}(s) \end{pmatrix} ds \tag{42}$$

because $(d/dt)[\mathbf{M}(t)\mathbf{M}^{-1}(t)] = 0$ and

$$(d/dt)[\mathbf{M}(t)\mathbf{M}^{-1}(t)] = \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) + \mathbf{M}(t)(d/dt)\mathbf{M}^{-1}(t)$$

which together imply the identity for t-dependent matrices

$$(d/dt)\mathbf{M}^{-1}(t) = -\,\mathbf{M}^{-1}(t)\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) \tag{43}$$

Therefore, (41) and (42) provide the two-component analog to (8).

To get the conditional probability distribution determined by (33) and denoted by $P(y, p, t)$ with the initial condition $P(y, p, 0) = \delta(y - y(0))$ $\delta(p - p(0))$, van Kampen's lemma is used again. Beginning with a phase space density $\rho(y, p, t)$, the continuity equation in this case may be written as

$$\frac{\partial}{\partial t}\rho(y, p, t) = -\frac{\partial}{\partial \mathbf{x}} \cdot [\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\mathbf{x}\rho(\mathbf{x}, t)] - \frac{\partial}{\partial \mathbf{x}} \cdot [\tilde{G}(t)\rho(\mathbf{x}, t)] \quad (44)$$

in which $\mathbf{x}$ denotes the two-component vector $\binom{y}{p}$, and $(\partial/\partial\mathbf{x}) \cdot$ denotes the two-dimensional divergence operator. Equation (44) is a multiplicative stochastic process, and van Kampen's lemma asserts that $\langle\rho(\mathbf{x}, t)\rangle \equiv P(\mathbf{x}, t)$. To get $\langle\rho(\mathbf{x}, t)\rangle$, a sequence of identities that parallels the development in Section 3 will be used.

Define $\phi(\mathbf{x}, t)$ by

$$\rho(\mathbf{x}, t) = \left\{\underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]\right\}\phi(\mathbf{x}, t) \quad (45)$$

in which $\underset{\leftarrow}{T}\exp[\cdots]$ denotes the time-ordered exponential,[4] which is required here because the operator $(\partial/\partial\mathbf{x}) \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}$ does not commute with itself at different times. This is the primary complication a multicomponent process creates. If $\mathbf{B}(t)$ is an arbitrary time-dependent operator that does not commute with itself at different times, then the inverse of $\underset{\leftarrow}{T}\exp[\int_0^t \mathbf{B}(s)\, ds]$ is given by $\underset{\rightarrow}{T}\exp[-\int_0^t \mathbf{B}(s)\, ds]$, in which it must be noticed that the time ordering is in the reversed sense. Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = &-\left\{\underset{\rightarrow}{T}\exp\left[\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]\right\}\frac{\partial}{\partial \mathbf{x}} \cdot \tilde{G}(t) \\ &\times \left\{\underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]\right\}\phi(\mathbf{x}, t) \end{aligned} \quad (46)$$

Identity 3:

$$\begin{aligned}&\left\{\underset{\rightarrow}{T}\exp\left[\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]\right\}\frac{\partial}{\partial \mathbf{x}} \cdot \tilde{G}(t) \\ &\quad\times \underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right] \\ &= \frac{d}{dt}\frac{\partial}{\partial \mathbf{x}} \cdot \int_0^t \mathbf{M}^{-1}(t)\mathbf{M}(t-s)\begin{pmatrix}0\\ \tilde{f}(s)\end{pmatrix} ds \end{aligned} \quad (47)$$

Proof:

$$\left\{\underset{\rightarrow}{T}\exp\left[\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]\right\} \frac{\partial}{\partial \mathbf{x}} \cdot \tilde{G}(t)$$

$$\times \underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]$$

$$= \underset{\rightarrow}{T}\exp\left\{\int_0^t \left[\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}, \cdot\right] ds\right\} \frac{\partial}{\partial \mathbf{x}} \cdot \tilde{G}(t) \tag{48}$$

where the right-hand side of this equation contains the time-ordered exponential of a commutator operator denoted by $[(\partial/\partial \mathbf{x}) \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}, \cdot]$. The first-order term in this exponential contains

$$\left[\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}, \frac{\partial}{\partial \mathbf{x}} \cdot \tilde{G}(t)\right] = -\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\tilde{G}(t) \tag{49}$$

This makes it clear that each higher order commutator will contain only the first-order $(\partial/\partial \mathbf{x}) \cdot$ factor, although the time-dependent matrix product will become increasingly complicated. However, the result, to all orders, may be expressed by the identity

$$\underset{\rightarrow}{T}\exp\left\{\int_0^t \left[\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}, \cdot\right] ds\right\} \frac{\partial}{\partial \mathbf{x}} \cdot \tilde{G}(t)$$

$$= \frac{\partial}{\partial \mathbf{x}} \cdot \left\{\underset{\rightarrow}{T}\exp\left[-\int_0^t \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\, ds\right]\right\}\tilde{G}(t) \tag{50}$$

Note that even though (48) begins with both time-ordering senses represented, only one sense of time ordering is required for the commutator exponentials.

The expression $\underset{\rightarrow}{T}\exp[-\int_0^t \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\, ds]$ can be simplified considerably by noting that at $t = 0$ it gives the 2×2 identity matrix which is identical with both $\mathbf{M}(0)$ and $\mathbf{M}^{-1}(0)$, and it satisfies the first-order differential equation

$$\frac{d}{dt}\underset{\rightarrow}{T}\exp\left[-\int_0^t \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\, ds\right]$$

$$= -\left\{\underset{\rightarrow}{T}\exp\left[-\int_0^t \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\, ds\right]\right\}\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) \tag{51}$$

wherein the derivative of the exponent appears on the right because the exponential is ordered to the right as indicated. This is identical with Eq. (43) for the derivative of $\mathbf{M}^{-1}(t)$, and it was already noted that the ordered exponential agrees with $\mathbf{M}^{-1}(0)$ at $t = 0$. Therefore

$$\underset{\rightarrow}{T}\exp\left[-\int_0^t \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\, ds\right] = \mathbf{M}^{-1}(t) \tag{52}$$

Using (52) in (50) and using (42) gives the right-hand side of (47).

Therefore, (46) becomes

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = -\frac{d}{dt}\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t \mathbf{M}^{-1}(t)\mathbf{M}(t-s)\begin{pmatrix}0\\ \tilde{f}(s)\end{pmatrix} ds\,\phi(\mathbf{x}, t) \tag{53}$$

Because $\langle \tilde{f}(t)\rangle = 0, \tilde{f}(t)$ is Gaussian, and the differential operator

$$\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t \mathbf{M}^{-1}(t)\mathbf{M}(t-s)\begin{pmatrix}0\\ \tilde{f}(s)\end{pmatrix} ds$$

commutes with itself at different times, it follows that the second cumulant is exact in this case, as in the case of free Brownian motion, which was discussed earlier, and it implies

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle\phi(\mathbf{x}, t)\rangle\\
&\quad= \int_0^t ds\left\langle\frac{d}{dt}\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t dt'\mathbf{M}^{-1}(t)\mathbf{M}(t-t')\begin{pmatrix}0\\ \tilde{f}(t')\end{pmatrix}\right.\\
&\qquad\left.\times\frac{d}{ds}\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t ds'\,\mathbf{M}^{-1}(s)\mathbf{M}(s-s')\begin{pmatrix}0\\ \tilde{f}(s')\end{pmatrix}\right\rangle\langle\phi(\mathbf{x}, t)\rangle\\
&\quad= \left\langle\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t ds'\,\mathbf{M}^{-1}(t)\mathbf{M}(t-s')\begin{pmatrix}0\\ \tilde{f}(s')\end{pmatrix}\right.\\
&\qquad\left.\times\frac{d}{dt}\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t dt'\,\mathbf{M}^{-1}(t)\mathbf{M}(t-t')\begin{pmatrix}0\\ \tilde{f}(t')\end{pmatrix}\right\rangle\langle\phi(\mathbf{x}, t)\rangle
\end{aligned}\tag{54}$$

The second equality follows from the fact that the integrand in the first is an exact differential with respect to s-integration. The last equality of (54) can be treated in parallel with (19) to give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\langle\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t ds'\,\mathbf{M}^{-1}(t)\mathbf{M}(t-s')\begin{pmatrix}0\\ \tilde{f}(s')\end{pmatrix}\right.\\
&\qquad\left.\times\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t dt'\,\mathbf{M}^{-1}(t)\mathbf{M}(t-t')\begin{pmatrix}0\\ \tilde{f}(t')\end{pmatrix}\right\rangle\langle\phi(\mathbf{x}, t)\rangle\\
&\quad= \frac{1}{2}K_{\mathrm{B}}Tm\frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}}\cdot\int_0^t ds'\int_0^t dt'\,\mathbf{M}^{-1}(t)\mathbf{M}(t-s')\right.\\
&\qquad\left.\times\,\mathbf{D}(s'-t')\mathbf{M}^{\dagger}(t-t')[\mathbf{M}^{-1}(t)]^{\dagger}\cdot\frac{\partial}{\partial \mathbf{x}}\right\}\langle\phi(\mathbf{x}, t)\rangle
\end{aligned}\tag{55}$$

in which the dagger denotes the transpose of a matrix and $\mathbf{D}(s'-t')$ is defined by

$$K_{\mathrm{B}}Tm\mathbf{D}(s-t) \equiv \left\langle\begin{pmatrix}0\\ \tilde{f}(s)\end{pmatrix}(0\quad \tilde{f}(t))\right\rangle = \begin{pmatrix}0 & 0\\ 0 & K_{\mathrm{B}}Tm\beta(s-t)\end{pmatrix} \tag{56}$$

Define $\mathbf{A}(t)$ by

$$\mathbf{A}(t) \equiv \int_0^t ds \int_0^t dr\, \mathbf{M}(s)\mathbf{D}(r - s)\mathbf{M}^{\dagger}(r) \tag{57}$$

Equation (54) can now be written in the form

$$\frac{\partial}{\partial t}\langle\phi(\mathbf{x}, t)\rangle = \frac{1}{2} K_{\mathrm{B}}Tm \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)\mathbf{A}(t)[\mathbf{M}^{-1}(t)]^{\dagger} \cdot \frac{\partial}{\partial \mathbf{x}}\right\}\langle\phi(\mathbf{x}, t)\rangle \tag{58}$$

which parallels the final line of (19).

To continue the parallels, the following equation for $\dot{\mathbf{M}}(t)$ is required, in which $\mathbf{\Omega} \equiv \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$:

$$(d/dt)\mathbf{M}(t) = \mathbf{M}(t)\mathbf{\Omega} - \int_0^t \mathbf{M}(s)\mathbf{D}(t - s)\, ds \tag{59}$$

This equation can be derived algebraically by looking at the Laplace transforms of each term, which gives

$$\dot{\mathbf{M}}(t) \to z\hat{\mathbf{M}}(z) - \mathbf{M}(0) = \begin{pmatrix} -\omega^2 & \omega z \\ -\omega z & -\omega^2 - z\hat{\beta}(z) \end{pmatrix}\{z[z + \hat{\beta}(z)] + \omega^2\}^{-1} \tag{60}$$

$$\mathbf{M}(t)\mathbf{\Omega} \to \hat{\mathbf{M}}(z)\mathbf{\Omega} = \begin{pmatrix} -\omega^2 & \omega[z + \hat{\beta}(z)] \\ -\omega z & -\omega^2 \end{pmatrix}\{z[z + \hat{\beta}(z)] + \omega^2\}^{-1}$$

$$- \int_0^t \mathbf{M}(s)\mathbf{D}(t - s)\, ds \tag{61}$$

$$\to -\hat{\mathbf{M}}(z)\hat{\mathbf{D}}(z) = \begin{pmatrix} 0 & -\omega\hat{\beta}(z) \\ 0 & -z\hat{\beta}(z) \end{pmatrix}\{z[z + \hat{\beta}(z)] + \omega^2\}^{-1} \tag{62}$$

in which the right-hand side of (35) is used for $\hat{\mathbf{M}}(z)$ and in which

$$\hat{\mathbf{D}}(z) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\beta}(z) \end{pmatrix}$$

has been used in (62).

With (57) and (59), it is possible to get a much more revealing expression for $\mathbf{A}(t)$:

$$(d/dt)\mathbf{A}(t) = \mathbf{M}(t)\int_0^t dr\, \mathbf{D}(r - t)\mathbf{M}^{\dagger}(r) + \int_0^t \mathbf{M}(s)\mathbf{D}(t - s)\, ds\, \mathbf{M}^{\dagger}(t) \tag{63}$$

From (59) it follows that

$$(d/dt)\mathbf{M}^{\dagger}(t) = -\mathbf{\Omega}\mathbf{M}^{\dagger}(t) - \int_0^t \mathbf{D}(t - s)\mathbf{M}^{\dagger}(s)\, ds \tag{64}$$

because $\mathbf{\Omega}$ is antisymmetric and $\mathbf{D}$ is symmetric as matrices. $\mathbf{D}(t-s)$ is also symmetric as a function of s and t. Therefore, (59) and (64) provide

$$\int_0^t \mathbf{D}(t-r)\mathbf{M}^\dagger(r)\,dr = -\mathbf{\Omega}\mathbf{M}^\dagger(t) - \dot{\mathbf{M}}^\dagger(t) \tag{65}$$

and

$$\int_0^t \mathbf{M}(s)\mathbf{D}(t-s)\,ds = \mathbf{M}(t)\mathbf{\Omega} - \dot{\mathbf{M}}(t) \tag{66}$$

When (65) and (66) are placed in (63) the result is

$$\begin{aligned}\dot{\mathbf{A}}(t) &= \mathbf{M}(t)[-\mathbf{\Omega}\mathbf{M}^\dagger(t) - \dot{\mathbf{M}}^\dagger(t)] + [\mathbf{M}(t)\mathbf{\Omega} - \dot{\mathbf{M}}(t)]\mathbf{M}^\dagger(t) \\ &= -(d/dt)[\mathbf{M}(t)\mathbf{M}^\dagger(t)]\end{aligned} \tag{67}$$

This equation is trivially integrated with the initial condition $\mathbf{A}(0) = 0$ and gives

$$\mathbf{A}(t) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \mathbf{M}(t)\mathbf{M}^\dagger(t) \tag{68}$$

in striking parallel with (22). This means that (58) can be written as

$$\frac{\partial}{\partial t}\langle\phi(\mathbf{x},t)\rangle = \frac{1}{2}K_B Tm\frac{d}{dt}\left\{\frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger\cdot\frac{\partial}{\partial\mathbf{x}}\right\}\langle\phi(\mathbf{x},t)\rangle \tag{69}$$

Therefore, (45) implies

$$\begin{aligned}\frac{\partial}{\partial t}\langle\rho(\mathbf{x},t)\rangle &= -\frac{\partial}{\partial\mathbf{x}}\cdot[\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\mathbf{x}\langle\rho(\mathbf{x},t)\rangle] \\ &\quad + \underset{\leftarrow}{T}\exp\left[-\int_0^t\frac{\partial}{\partial\mathbf{x}}\cdot\dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\,ds\right]\frac{\partial}{\partial t}\langle\phi(\mathbf{x},t)\rangle \\ &= -\frac{\partial}{\partial\mathbf{x}}\cdot[\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\mathbf{x}\langle\rho(\mathbf{x},t)\rangle] \\ &\quad + \frac{1}{2}K_B Tm\,\underset{\leftarrow}{T}\exp\left[-\int_0^t\frac{\partial}{\partial\mathbf{x}}\cdot\dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\,ds\right] \\ &\quad \times\frac{d}{dt}\left\{\frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger\cdot\frac{\partial}{\partial\mathbf{x}}\right\} \\ &\quad \cdot\left\{\underset{\rightarrow}{T}\exp\left[\int_0^t\frac{\partial}{\partial\mathbf{x}}\cdot\dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\,ds\right]\right\}\langle\rho(\mathbf{x},t)\rangle\end{aligned} \tag{70}$$

Identity 4:

$$\underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]$$
$$\times \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger \cdot \frac{\partial}{\partial \mathbf{x}}\right\} \cdot \underset{\rightarrow}{T}\exp\left[\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]$$
$$= -2\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) \cdot \frac{\partial}{\partial \mathbf{x}} \tag{71}$$

Proof. Closely related to (48) is the relation

$$\underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]$$
$$\times \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger \cdot \frac{\partial}{\partial \mathbf{x}}\right\} \cdot \underset{\rightarrow}{T}\exp\left[\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\, ds\right]$$
$$= \underset{\leftarrow}{T}\exp\left\{-\int_0^t \left[\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x},\ \cdot\ \right] ds\right\}$$
$$\times \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger \cdot \frac{\partial}{\partial \mathbf{x}}\right\} \tag{72}$$

which involves the ordered exponential of a commutator operator. The first-order term in this exponential contains

$$-\left[\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}, \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger \cdot \frac{\partial}{\partial \mathbf{x}}\right\}\right]$$
$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger \cdot \frac{\partial}{\partial \mathbf{x}}\right.$$
$$\left. + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger[\mathbf{M}^{-1}(s)]^\dagger\dot{\mathbf{M}}^\dagger(s) \cdot \frac{\partial}{\partial \mathbf{x}}\right\}$$
$$\equiv \frac{d}{dt}\left(\frac{\partial}{\partial \mathbf{x}} \cdot \left\{\dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s), \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^\dagger\right\}_\dagger \cdot \frac{\partial}{\partial \mathbf{x}}\right) \tag{73}$$

where the second equality defines the operator $\{\mathbf{B}(s), \cdot\}_\dagger$ to be given by

$$\{\mathbf{B}(s), \cdot\}_\dagger \mathbf{E} \equiv \mathbf{B}(s)\mathbf{E} + \mathbf{E}\mathbf{B}^\dagger(s) \tag{74}$$

in which $\mathbf{B}(s)$ and $\mathbf{E}$ are arbitrary matrices. Note that the result in (73) contains exactly two orders of $\partial/\partial\mathbf{x}$. Therefore, the entire exponential on the right-hand side of (72) can be written as

$$\underset{\leftarrow}{T}\exp\left\{-\int_0^t \left[\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x},\ \cdot\ \right]ds\right\} \frac{d}{dt}\left\{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^{\dagger} \cdot \frac{\partial}{\partial \mathbf{x}}\right\}$$
$$= \frac{\partial}{\partial \mathbf{x}} \cdot \left(\underset{\leftarrow}{T}\exp\left[\int_0^t \{\dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s),\ \cdot\}_{\dagger}\, ds\right] \frac{d}{dt}\{\mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^{\dagger}\}\right) \cdot \frac{\partial}{\partial \mathbf{x}} \tag{75}$$

This relation is closely related to (50), and the exponential on the right-hand side may be simplified considerably, in parallel with (52),

$$\underset{\leftarrow}{T}\exp[\int_0^t \{\dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s),\ \cdot\}_{\dagger}\, ds] = \mathbf{M}(t) \cdot \mathbf{M}^{\dagger}(t) \tag{76}$$

because both sides agree at $t = 0$ and they both satisfy the first-order differential equation

$$(d/dt)\mathbf{U}(t) = \{\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t), \mathbf{U}(t)\}_{\dagger} \tag{77}$$

as is readily verified. Using (43) and (76) in (75) gives

$$\underset{\leftarrow}{T}\exp\left[\int_0^t \{\mathbf{M}(s)\mathbf{M}^{-1}(s),\ \cdot\}_{\dagger}\, ds\right] \frac{d}{dt}\{\mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^{\dagger}\}$$
$$= -\mathbf{M}(t)\{\mathbf{M}^{-1}(t)\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^{\dagger}$$
$$+ \mathbf{M}^{-1}(t)[\mathbf{M}^{-1}(t)]^{\dagger}\dot{\mathbf{M}}^{\dagger}(t)[\mathbf{M}^{-1}(t)]^{\dagger}\}\mathbf{M}^{\dagger}(t)$$
$$= -\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) - [\mathbf{M}^{-1}(t)]^{\dagger}\dot{\mathbf{M}}^{\dagger}(t) \tag{78}$$

It is also so that

$$\frac{\partial}{\partial \mathbf{x}} \cdot \{\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) + [\mathbf{M}^{-1}(t)]^{\dagger}\dot{\mathbf{M}}^{\dagger}(t)\} \cdot \frac{\partial}{\partial \mathbf{x}} = 2\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) \cdot \frac{\partial}{\partial \mathbf{x}} \tag{79}$$

Using (79) and (78) in (75) and (72) justifies Identity 4.

Consequently, (70) becomes

$$\frac{\partial}{\partial t}\langle\rho(\mathbf{x}, t)\rangle = -\frac{\partial}{\partial \mathbf{x}} \cdot [\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\mathbf{x}\langle\rho(\mathbf{x}, t)\rangle]$$
$$- K_{\mathrm{B}}Tm\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) \cdot \frac{\partial}{\partial \mathbf{x}}\langle\rho(\mathbf{x}, t\rangle \tag{80}$$

Using van Kampen's lemma provides the Fokker–Planck equation for this Brownian oscillator, which is

$$\frac{\partial}{\partial t}P(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}} \cdot [\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\mathbf{x}P(\mathbf{x}, t)]$$
$$- K_{\mathrm{B}}Tm\frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t) \cdot \frac{\partial}{\partial \mathbf{x}}P(\mathbf{x}, t) \tag{81}$$

Notice that the matrix coefficients $\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)$ are t dependent, so that (81) describes a *nonstationary Gaussian* process, which is consequently also a *nonstationary Markovian* process.

It may be verified by substitution in (81) that the solution satisfying the initial condition $P(y, p, 0) = \delta(y - y(0))\,\delta(p - p(0))$ is

$$P(y, p, t) = (2\pi K_B Tm)^{-1}\{\det[\mathbf{A}(t)]\}^{-1/2} \times \exp\left\{-\frac{1}{2K_B Tm}[\mathbf{x} - \mathbf{M}(t)\mathbf{x}(0)]^{\dagger}\mathbf{A}^{-1}(t)[\mathbf{x} - \mathbf{M}(t)\mathbf{x}(0)]\right\} \quad (82)$$

in which $\mathbf{A}(t)$ is given by (68) and $\mathbf{x}(0) = \begin{pmatrix} y(0) \\ p(0) \end{pmatrix}$. The notation $\det[\mathbf{A}(t)]$ denotes the determinant of the matrix $\mathbf{A}(t)$. Equation (82) provides the complete stochastic description of the "non-Markovian" Brownian oscillator. Note that the result is in fact a nonstationary, Gaussian, *Markov* process.

5. NON-GAUSSIAN PROCESSES

In order for the setting for the non-Gaussian case to be as general as possible, it should be noted that most of the details involved in the analysis of the "non-Markovian" Brownian oscillator did not really depend upon the explicit 2×2 matrices used but instead depended upon only quite general matrix algebraic properties, which apply equally well to systems with more than two components. Specifically, Eq. (33) is a special case of

$$(d/dt)\mathbf{x}(t) = \mathbf{\Omega}\mathbf{x}(t) - \int_0^t \mathbf{D}(t - s)\mathbf{x}(s)\,ds + \tilde{\mathbf{f}}(t) \quad (83)$$

in which $\mathbf{x}(t)$ denotes an n-component vector, $\mathbf{\Omega}$ denotes an $n \times n$ antisymmetric matrix, and $\mathbf{D}(t - s)$ denotes a symmetric, $n \times n$ matrix "memory kernel." The process is driven by a stochastic vector force $\tilde{\mathbf{f}}(t)$, which will generally have n components. The fluctuation-dissipation relation is now

$$\langle \tilde{\mathbf{f}}(t)\tilde{\mathbf{f}}(s)^{\dagger}\rangle = K_B T\mathbf{D}(t - s) \quad (84)$$

The Adelman transformation converts (83) into

$$\frac{d}{dt}\mathbf{x}(t) = \dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\mathbf{x}(t) + \mathbf{M}(t)\frac{d}{dt}\int_0^t \mathbf{M}^{-1}(t)\mathbf{M}(t - s)\tilde{\mathbf{f}}(s)\,ds \quad (85)$$

in which $\mathbf{M}(t)$ is defined through its Laplace transform $\hat{\mathbf{M}}(z)$, which is given by

$$\hat{\mathbf{M}}(z) = [z\mathbf{1} - \mathbf{\Omega} + \hat{\mathbf{D}}(z)]^{-1} \quad (86)$$

where $\mathbf{1}$ is the $n \times n$ identity matrix and $\hat{\mathbf{D}}(z)$ is the Laplace transform of $\mathbf{D}(t - s)$. It is now possible to proceed in the Gaussian case from (44) to

(58) without alteration because those steps in the 2×2 case did not use the explicit 2×2 nature of the matrices involved. All that must be shown is that Eq. (59) can be justified for the $n \times n$ case because Eqs. (60)–(62) explicitly involve the 2×2 case matrices. The analogs of (60)–(62) are

$$\begin{aligned}\dot{\mathbf{M}}(t) \to z\hat{\mathbf{M}}(z) - \mathbf{1} &= z[z\mathbf{1} - \mathbf{\Omega} + \hat{\mathbf{D}}(z)]^{-1} - \mathbf{1} \\ &= [z\mathbf{1} - \mathbf{\Omega} + \hat{\mathbf{D}}(z)]^{-1}[\mathbf{\Omega} - \hat{\mathbf{D}}(z)] \end{aligned} \tag{87}$$

$$\mathbf{M}(t)\mathbf{\Omega} \to \hat{\mathbf{M}}(z)\mathbf{\Omega} = [z\mathbf{1} - \mathbf{\Omega} + \mathbf{D}(z)]^{-1}\mathbf{\Omega} \tag{88}$$

$$-\int_0^t \mathbf{M}(s)\mathbf{D}(t-s)\,ds \to -\hat{\mathbf{M}}(z)\hat{\mathbf{D}}(z) = -[z\mathbf{1} - \mathbf{\Omega} + \hat{\mathbf{D}}(z)]^{-1}\hat{\mathbf{D}}(z) \tag{89}$$

Therefore, it clearly follows that

$$(d/dt)\mathbf{M}(t) = \mathbf{M}(t)\mathbf{\Omega} - \int_0^t \mathbf{M}(s)\mathbf{D}(t-s)\,ds \tag{90}$$

Now one may proceed from (63) all the way to (82) without alteration in the argument, provided that the left-hand side of (82) is written as $P(\mathbf{x}, t)$, in which $\mathbf{x}$ denotes an n-component vector, and the right-hand side contains $(2\pi K_B T)^{-n/2}$.

For the non-Gaussian case, the critical step is the step from Eq. (53) to Eq. (54), or their analogs in the n-component case. The second cumulant will no longer be exact and higher order cumulants will be required. Explicit combinatorial formulas exist[4] for all of the higher order cumulants, so that it is possible to write out explicit formulas that generalize Eq. (54). Because $\langle \tilde{\mathbf{f}}(t) \rangle = 0$, only even-order cumulants will be nonvanishing. Because the differential operator

$$-\frac{d}{dt}\frac{\partial}{\partial \mathbf{x}} \cdot \int_0^t \mathbf{M}^{-1}(t)\mathbf{M}(t-s)\tilde{\mathbf{f}}(s)\,ds$$

commutes with itself at different times, only ordinary cumulants are required and ordered cumulants simply reduce to ordinary cumulants. Whereas the second cumulant that occurs in (54) is second order in $\partial/\partial \mathbf{x}$, the nth cumulant will be nth order in $\partial/\partial \mathbf{x}$, as was suggested by Adelman.[1] The analog of (54) will have the form

$$\frac{\partial}{\partial t}\langle \phi(\mathbf{x}, t) \rangle = \sum_{n=1}^{\infty} \mathbf{G}^{(2n)}(t)\langle \phi(\mathbf{x}, t) \rangle \tag{91}$$

in which $\mathbf{G}^{(2n)}(t)$ denotes the $2n$th cumulant and $\mathbf{G}^{(2)}(t)$ is identically given by (54). In order to subsequently get the equation for $\langle \rho(\mathbf{x}, t) \rangle$, it will be necessary to compute the analog of Identity 4 for the higher cumulants

$$\left\{ \underset{\leftarrow}{T}\exp\left[-\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\,ds \right] \right\} \mathbf{G}^{(2n)}(t)\, \underset{\rightarrow}{T}\exp\left[\int_0^t \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}\,ds \right]$$

Because $\mathbf{G}^{(2n)}(t)$ is of order $2n$ in $\partial/\partial\mathbf{x}$, the expression above will also turn out to be of order $2n$ in $\partial/\partial\mathbf{x}$ because the exponentials contain the combination

$$(\partial/\partial\mathbf{x}) \cdot \dot{\mathbf{M}}(s)\mathbf{M}^{-1}(s)\mathbf{x}$$

The final result will always involve infinitely many higher cumulants, so that approximations through truncation of the cumulant series will be required. Closed-form solutions to these generalized Fokker–Planck equations that possess higher than second-order derivatives in $\mathbf{x}$ are not known for either the entire cumulant series or for its truncations. However, the generalized Fokker–Planck equations readily generate equations for the moments of the original stochastic process $\mathbf{x}$, and these moment equations usually permit elementary solutions.

It must be emphasized that for the Gaussian case, Eq. (29) for the free Brownian motion has been obtained by several others[3] as well as by Adelman,[(1)] and that Eq. (81), in a somewhat different but equivalent form, has been obtained by Adelman.[(1)] However, the methods used here also readily develop into a description of non-Gaussian processes. Although that description is perhaps combinatorially complicated, it is nevertheless known explicitly.[(4)] The virtues of the multiplicative stochastic process approach, with its use of ordered and ordinary cumulants, may be exhibited in many other contexts as well. Moreover, the use of time ordering and related operator calculus techniques, which usually appear only in quantum mechanical contexts, should also be noted. These methods enjoy a broad applicability in classical contexts, as has been exhibited here.

It is to be especially noted that for the Gaussian cases the Adelman transformation, in its extended, stochastic form, results in a nonstationary Fokker–Planck equation, which describes a *nonstationary*, *Gaussian*, *Markovian* process. This is in spite of the fact that projection operator techniques that give rise to the "memory kernel" equations have led many authors[(1)] and other workers to refer to those processes as *non-Markovian*. However, all the memory kernel really does is produce a *nonstationary*, *Markovian* process.[(15)] The methods of multiplicative stochastic process analysis make this fact especially transparent and suggest that these methods are more fundamental. The point will be reinforced in other contexts in subsequent papers.

REFERENCES

1. S. A. Adelman, *J. Chem. Phys.* **64**:124 (1976).
2. R. Kubo, *J. Phys. Soc. Japan* **17**:1100 (1962).
3. N. G. van Kampen, *Physica* **74**:215, 239 (1974).

---

[3] See Adelman,[(1)] particularly the note added in proof.

4. R. F. Fox, *J. Math. Phys.* **17**:1148 (1976).
5. R. Zwanzig, *Phys. Rev.* **124**:983 (1961).
6. H. Mori, *Prog. Theor. Phys.* **33**:423 (1965).
7. R. H. Terwiel, *Physica* **74**:248 (1974).
8. B. Yoon, J. M. Deutsch, and J. H. Freed, *J. Chem. Phys.* **62**:4687 (1975).
9. N. G. van Kampen, *Physics Reports* **24C**:171 (1976).
10. R. F. Fox, *J. Math. Phys.* **15**:1918 (1974).
11. R. Kubo, *Rep. Prog. Theor. Phys.* **29**:255 (1966).
12. M. C. Wang and G. E. Uhlenbeck, *Rev. Mod. Phys.* **17**:323 (1945).
13. J. R. Dorfman, in *Fundamental Problems in Statistical Mechanics III*, E. G. D. Cohen, ed., North-Holland, Amsterdam (1975), Section 5.
14. G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.* **36**:823 (1930).
15. L. Arnold, *Stochastic Differential Equations, Theory and Applications*, Wiley–Interscience, New York (1974), Chapters 2 and 8.